Sound propagation in a fluid flowing through an attenuating duct

By D. C. PRIDMORE-BROWN*
Department of Mathematics, University of Manchester

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SUMMARY

A study is made of the propagation of sound in both a constant gradient shear flow and a turbulent shear flow above a flat surface. Curves are presented showing how, in the case of downstream propagation, the flow gradient tends to channel the sound energy into a narrow layer next to the wall. These results are used in estimating the effect of a flow on the attenuation of sound in a duct with absorbing side walls.

1. INTRODUCTION

Undesired sound is frequently attenuated by having it pass through ducts with absorbing side walls. If the absorptive properties of the walls are described by a normal impedance, and if there is no air flow in the duct, the resulting attenuation is easily predicted on the basis of existing theory (see, for example, Morse 1939). In many cases, however (e.g. in ventilating ducts or the exhaust ducts in certain wind tunnels or jet-engine test cells), there is also a flow of air through the duct which may affect the rate of sound attenuation. Downstream, for instance, the velocity gradients in the boundary layers might be expected to direct the sound into the absorbing walls and so increase the attenuation.

A particularly simple case to study is that of sound propagating in a fluid which is flowing between two parallel walls. Here the flow velocity can be assumed to be approximately constant throughout a central region between the walls and to fall to zero within the two boundary layers. Since the flow profile will be symmetric about a centre line drawn midway between the walls, it is then sufficient to consider just half the region, i.e. that between the centre line and one wall. On a ray acoustics picture it is clear that sound rays propagating upstream in one of the boundary layers will be bent away from the wall, while rays propagating downstream will be bent towards it, and in certain cases may be expected to suffer repeated successive reflections in the wall surface. A ray treatment is not, however, generally adequate for ascertaining the actual sound pressure distribution across such a shear layer owing to the tendency, pronounced at lower frequencies, for the acoustic energy to diffuse away from regions where it is concentrated. It is therefore desirable to adopt a wave treatment from the outset.

* Now at Mechanical Engineering Department, Massachusetts Institute of Technology.

We shall confine our attention to the problem of a two-dimensional acoustic wave propagating downstream in a shear layer above a plane wall. Within the shear layer we shall consider only the lowest mode of propagation. Thus the boundary condition to be imposed on the sound wave will be that the normal component of the particle velocity (in a direction perpendicular to the wall) must vanish both at the wall surface and at the top of the shear layer, so that the wave matches smoothly on to a plane wave above the layer. First we study the simplest case of perfectly rigid side walls and a constant gradient of flow velocity, and then we extend the analysis to a turbulent flow profile in which the flow velocity increases as the one-seventh power of the distance from the wall. For both these cases representative curves are given showing the variation of sound pressure across the duct which is brought about by the presence of the flow gradient. Finally, we shall study the effect of the shear flow on the attenuation of the sound when the walls have a small admittance. In this case the flow has two contrary effects on the sound transmission. In the first place (for downstream propagation) it tends to increase the absorption by directing sound into the walls as previously pointed out. On the other hand, for wavelengths long compared to the shear layer thickness, this refraction by the flow gradient becomes quite negligible and may in fact be counterbalanced by the increase of intensity in the central region which results from the transport of acoustic energy by the flow.

2. The basic equations

We shall first derive the wave equation in a form suitable for studying the propagation of sound in a shear flow. If we consider the flow velocity V to lie in the x-direction and to be a function of y only, then, neglecting viscosity, we obtain the linearized Navier–Stokes equations for the conservation of mass and momentum in the form

$$\frac{\partial \rho}{\partial t} + V\frac{\partial \rho}{\partial x} + \rho_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} + v\frac{\partial V}{\partial y} + \frac{1}{\rho_0}\frac{\partial p}{\partial x} = 0, \tag{2}$$

$$\frac{\partial v}{\partial t} + V\frac{\partial v}{\partial x} + \frac{1}{\rho_0}\frac{\partial p}{\partial y} = 0. \tag{3}$$

Here ρ_0 represents the static density of the medium, which is assumed constant; the density fluctuations ρ, the sound pressure p, and the particle velocity components u and v are small quantities of the first order. If the further assumption is made that $p = c^2\rho$, that is, that the pressure and density are adiabatically related, then these equations yield

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = (1-M^2)\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} - \frac{2M}{c}\frac{\partial^2 p}{\partial x \partial t} + 2\rho_0 c\frac{dM}{dy}\frac{\partial v}{\partial x}, \tag{4}$$

where $M = V/c$.

We shall be interested in solutions to equation (4) of the form

$$\left.\begin{aligned} p &= e^{ik(\kappa x - ct)}F(\kappa, y), \\ v &= e^{ik(\kappa x - ct)}G(\kappa, y), \end{aligned}\right\} \tag{5}$$

where F and G are related by equation (3),

$$\frac{dF}{dy} = i\rho_0 ck(1-\kappa M)G. \tag{6}$$

Substituting these expressions into the differential equation (4), we obtain

$$\frac{d^2F}{dy^2} + \frac{2\kappa M}{1-\kappa M}\frac{dF}{dy} + k^2[(1-\kappa M)^2 - \kappa^2]F = 0, \tag{7}$$

where $M' = dM/dy$. This equation must be satisfied by the pressure amplitude F subject to the boundary conditions $dF/dy = 0$ at $y = 0$ and $y = L$.

Although the approximate method of solution we shall employ can be applied to a fairly wide class of velocity profiles, the analysis is considerably simplified by taking the flow to have a constant velocity gradient. Actually, the velocity of a turbulent fluid flowing over a flat plate increases approximately as the one-seventh power of the distance from the plate. We shall, however, first consider the simple case of a constant gradient before treating the physically more realistic situation described by the one-seventh power law.

3. The constant gradient boundary layer

For the constant gradient case we put $M' = k/\Lambda$ where Λ is a constant and introduce $\eta = \kappa^{-1} - M$ as the independent variable in (7), which then reduces to

$$\frac{d^2F}{d\eta^2} - \frac{2}{\eta}\frac{dF}{d\eta} + \Lambda^2\kappa^2(\eta^2 - 1)F = 0. \tag{8}$$

Approximate solutions to equation (8), valid asymptotically for large Λ, can be obtained by a method proposed by Langer (1937). These are

$$F = \eta s^{1/6} q^{-1/4} f(u), \tag{9}$$

where

$$q = \eta^2 - 1,$$

$$s = \int_1^\eta q^{1/2}\, d\eta = \tfrac{1}{2}\{\eta(\eta^2-1)^{1/2} - \cosh^{-1}\eta\},$$

$$u = (\tfrac{3}{2}\Lambda\kappa s)^{2/3},$$

and $f(u)$ is the general solution of

$$f''(u) + uf(u) = 0. \tag{9 a}$$

These solutions will, of course, break down at $\eta = 0$, i.e. when the flow velocity equals the propagation velocity, and hence cannot be used through this point. We shall accordingly assume always that the flow is subsonic and $\eta > 0$. The function f can be expressed either in terms of one-third order Bessel functions, namely $u^{1/2}J_{1/3}(\frac{2}{3}u^{3/2})$ and $u^{1/2}J_{-1/3}(\frac{2}{3}u^{3/2})$, or as Airy integrals, $Ai(-u)$ and $Bi(-u)$.

If we denote two independent solutions of (8) by F_1 and F_2, then the combination of these which satisfies the boundary conditions is

$$F = F_1 - RF_2,$$

where

$$R = \left(\frac{dF_1/dy}{dF_2/dy}\right)_{\text{boundary}},$$

and

$$R(y=0) = R(y=L). \tag{10}$$

In terms of Airy integrals we have

$$F = \eta s^{1/6} q^{-1/4}[Bi(-u) - R\,Ai(-u)], \tag{11}$$

and

$$R = \frac{\phi\, Bi(-u) - (\Lambda\kappa)^{2/3}\, Bi'(-u)}{\phi\, Ai(-u) - (\Lambda\kappa)^{2/3}\, Ai'(-u)}, \tag{12}$$

where

$$\phi = (12s)^{-2/3}[1 - 3(2-\eta^2)/(\eta q^{3/2})].$$

Equation (10) determines the eigenvalues of κ. We shall be interested in the smallest one of these, corresponding to the lowest mode of propagation in the duct. For particular values of Λ and kL the equation can be solved for κ by a method of successive approximation. It is clear on physical grounds that for downstream propagation the propagation velocity $c\kappa^{-1}$ must lie between c and $c(1+M_1)$, where M_1 is the value of M at $y = L$. When there is little refraction by the flow, i.e. for low frequencies or small flow velocities, $c\kappa^{-1}$ will lie nearly midway between these values. On the other hand, in cases where the refraction is large and most of the sound is propagating near the walls, then $c\kappa^{-1}$ will be nearer the smaller of these values. These considerations are a help in guessing a first value for κ.

For purposes of calculation the quantities s, u, ϕ occurring in (11) and (12) are most conveniently expressed as power series in the small quantity $\epsilon = \eta - 1 = \kappa^{-1} - M - 1$, namely,

$$\begin{aligned}
s &= \frac{2\surd 2}{3}\,\epsilon^{3/2}[1 + 0{\cdot}150\epsilon - 0{\cdot}134\epsilon^2 + \ldots],\\
u &= 2^{1/3}(\Lambda\kappa)^{2/3}\,\epsilon[1 + 0{\cdot}100\epsilon - 0{\cdot}012\epsilon^2 + \ldots],\\
\phi &= 0{\cdot}7143 - 0{\cdot}8938\epsilon + 0{\cdot}955\epsilon^2 + \ldots,\\
F &= 3^{-1/6}(1 + 0{\cdot}90\epsilon)[Bi(-u) - R\,Ai(-u)].
\end{aligned}$$

Figures 1 and 2 are plots of the sound pressure level in decibels (i.e. of $20\log_{10}(p/p_1)$) across one of the boundary layers in the duct as a function of the distance from the wall for different values of M_1, the Mach number at the top of the layer. In figure 1, $kL = 2\pi$, that is, the boundary layer thickness L is equal to a wavelength, while in figure 2, $kL = 20$. The shear flow is seen to have a much greater effect on the high frequencies

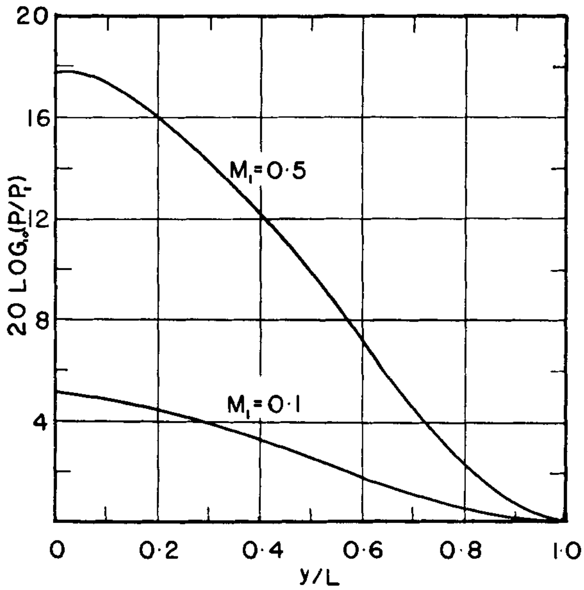

Figure 1. Sound pressure level in decibels (20 $\log_{10}(p/p_1)$) as a function of distance from the wall for a sound wave of frequency parameter $kL = 2\pi$ which propagates in a flow whose Mach number increases from 0 at the wall to M_1 at a distance L from the wall.

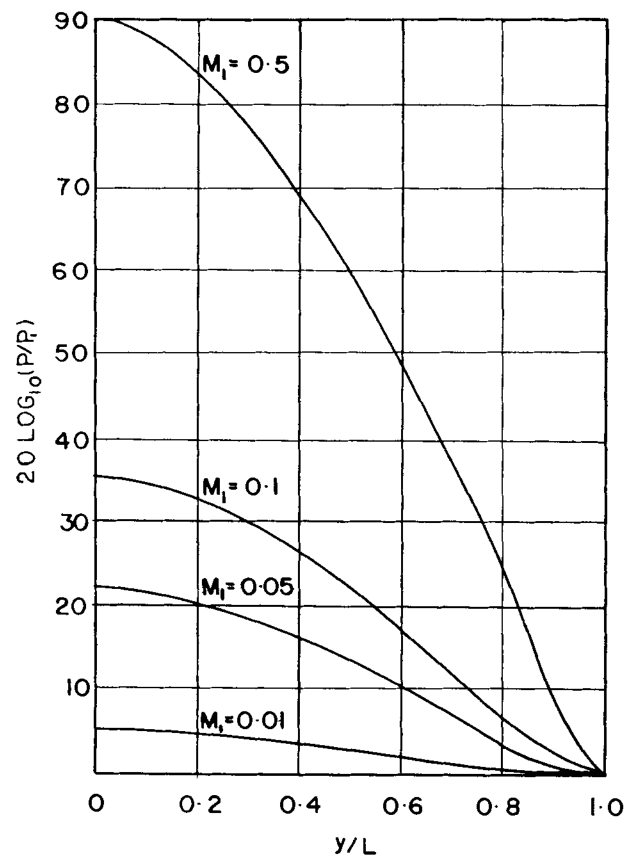

Figure 2. Sound pressure level in decibels as a function of distance from the wall when $kL = 20$.

than on the low; in fact, for the higher frequency ($kL = 20$) the theory predicts a pressure difference of as much as 90 decibels for a flow of Mach number 0·5. The Mach number dependence of the sound pressure profile is illustrated in figure 3, where the sound pressure difference in decibels across the shear layer has been plotted as a function of Mach number for the two cases $kL = 2\pi$ and $kL = 20$.

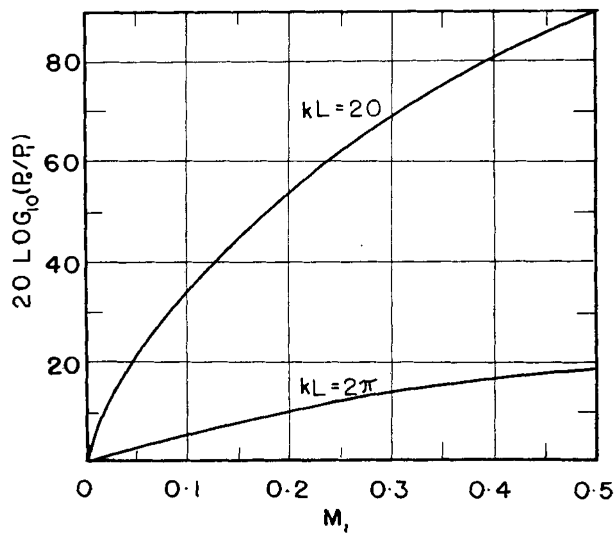

Figure 3. Sound pressure level difference in decibels across the shear layers of figures 1 and 2.

4. The turbulent boundary layer

We now consider a typical turbulent boundary layer in which the average flow velocity varies as the one-seventh power of the distance from the wall. Setting $M = M_1(y/L)^{1/7}$ in (7) and using M as the independent variable we obtain

$$\frac{d^2F}{dM^2} - \left(\frac{6}{M} - \frac{2}{\eta}\right)\frac{dF}{dM} + 49(kL)^2 M_1^{-14}\kappa^2 M^{12}(\eta^2 - 1)F = 0, \tag{13}$$

where, as before, $\eta = \kappa^{-1} - M$.

Langer's method can again be employed to give asymptotic solutions to this equation in the form

$$F = M^3\eta s^{1/6} q^{-1/4}[Bi(-u) - R\,Ai(-u)], \tag{14}$$

where

$$q = M^{12}(\eta^2 - 1), \qquad u = (\tfrac{3}{2}\Lambda\kappa s)^{2/3},$$

$$s = \int_1^\eta q^{1/2}\,d\eta, \qquad \Lambda = 7kLM_1^{-7}.$$

The boundary conditions are satisfied, as before, by requiring that R_0 be equal to R_1, the subscripts referring to values at $y = 0$ and $y = L$, respectively.

A difficulty now presents itself in determining the form of R at $y = 0$. This arises because the coefficient of dF/dM in equation (13) is singular at $y = 0$ (i.e. $M = 0$), and hence the equation does not satisfy the conditions of Langer's theory right at this point. Langer's solutions are, however, valid asymptotically ($\Lambda \to \infty$) in the neighbourhood of $y = 0$. Now equation (13) has only a regular singular point at $M = 0$, and hence its independent solutions can be found as Laurent series with indices 0 and 7. The first few terms of the general solution turn out to be

$$F = c_1(1 - \tfrac{1}{98}\Lambda^2(1-\kappa^2)M^{14} + \ldots) + \\ + c_2(M^7(1 - \tfrac{7}{4}\kappa M + \tfrac{7}{9}\kappa^2 M^2) - \tfrac{1}{294}\Lambda^2(1-\kappa^2)M^{21} + \ldots), \quad (15)$$

where c_1 and c_2 are constants. Remembering that y is proportional to M^7, we see that the condition $dF/dy = 0$ at $y = 0$ rules out the solution which starts as M^7. Hence we can replace the boundary condition at the wall by the condition that as $\Lambda \to \infty$ and $M \to 0$, equation (14) must tend to equation (15) with $c_2 = 0$. The calculation involves substituting the asymptotic forms of the Airy integrals into equation (14) and expanding the other quantities in powers of M. The result turns out to be that (15), with $c_2 = 0$, is asymptotically equivalent to

$$M^3\eta s^{1/6}q^{-1/4}[Bi(-u) + (Ai(-u_0)/Bi(-u_0))Ai(-u)],$$

where u_0 is u at $y = 0$. Thus we see that

$$R_0 = -Ai(-u_0)/Bi(-u_0). \quad (16)$$

At $y = L$ equation (13) is well behaved and hence R_1 has the form given in equation (12), where now

$$\phi = (12s)^{-2/3}\left[1 - \frac{3(2-\eta^2)s}{\eta(\eta^2-1)q^{1/2}}\right].$$

Again, for calculation it is useful to expand the various quantities appearing in (14) as series in $\epsilon = \eta - 1$ and $\beta = \epsilon/(\kappa^{-1} - 1)$. First,

$$s = \tfrac{2}{3}\surd 2(\kappa^{-1}-1)^6\epsilon^{3/2}(a_0 + a_1\epsilon + a_2\epsilon^2 + \ldots),$$

where

$$a_0 = 3(\tfrac{1}{3} - \tfrac{6}{5}\beta + \tfrac{15}{7}\beta^2 - \tfrac{20}{9}\beta^3 + \tfrac{15}{11}\beta^4 - \tfrac{6}{13}\beta^5 + \tfrac{1}{15}\beta^6),$$
$$a_1 = \tfrac{3}{4}(\tfrac{1}{5} - \tfrac{6}{7}\beta + \tfrac{15}{9}\beta^2 - \ldots + \tfrac{1}{17}\beta^6),$$
$$a_2 = -\tfrac{3}{32}(\tfrac{1}{6} - \tfrac{6}{9}\beta + \tfrac{15}{11}\beta^2 - \ldots + \tfrac{1}{19}\beta^6),$$

and so on. Secondly,

$$u = 2^{1/3}(\Lambda\kappa)^{2/3}(\kappa^{-1}-1)a_0^{2/3}(1 + c_1\epsilon + c_2\epsilon^2 + \ldots),$$

where

$$c_1 = \frac{2}{3}\frac{a_1}{a_0}, \qquad c_2 = \frac{2}{3}\frac{a_2}{a_0} - \frac{1}{9}\left(\frac{a_1}{a_0}\right)^2.$$

Thirdly,

$$\phi = 2^{-7/3}(\kappa^{-1}-1)a_0^{-2/3}\epsilon^{-1}(d_0 + d_1\epsilon + d_2\epsilon^2 + \ldots),$$

where

$$d_0 = 1 - b_0, \qquad d_1 = 0{\cdot}25 - 3{\cdot}5b_0 - b_1 - c_1(1 - b_0),$$

and
$$b_i = a_i/(1-\beta)^6.$$

Finally,
$$F = 3^{-1/6}(\kappa^{-1}-1)a_0^{1/6}(1+g_1\epsilon+g_2\epsilon^2+\ldots)\{Bi(-u)-R\,Ai(-u)\},$$
where
$$g_1 = 0{\cdot}875+0{\cdot}167\,\frac{a_1}{a_0},$$
$$g_2 = -0{\cdot}086+0{\cdot}146\,\frac{a_1}{a_0}+0{\cdot}167\,\frac{a_2}{a_0}-0{\cdot}07\left(\frac{a_1}{a_0}\right)^2.$$

As an aid in calculation the quantities $a_0^{2/3}$, $a_0^{1/6}$, $-c_1$ and g_1 have been plotted in figure 4 as functions of $\beta = 1-M(y)/(\kappa^{-1}-1)$. Note that in order to evaluate F for a given set of values of the parameters M_1 and kL,

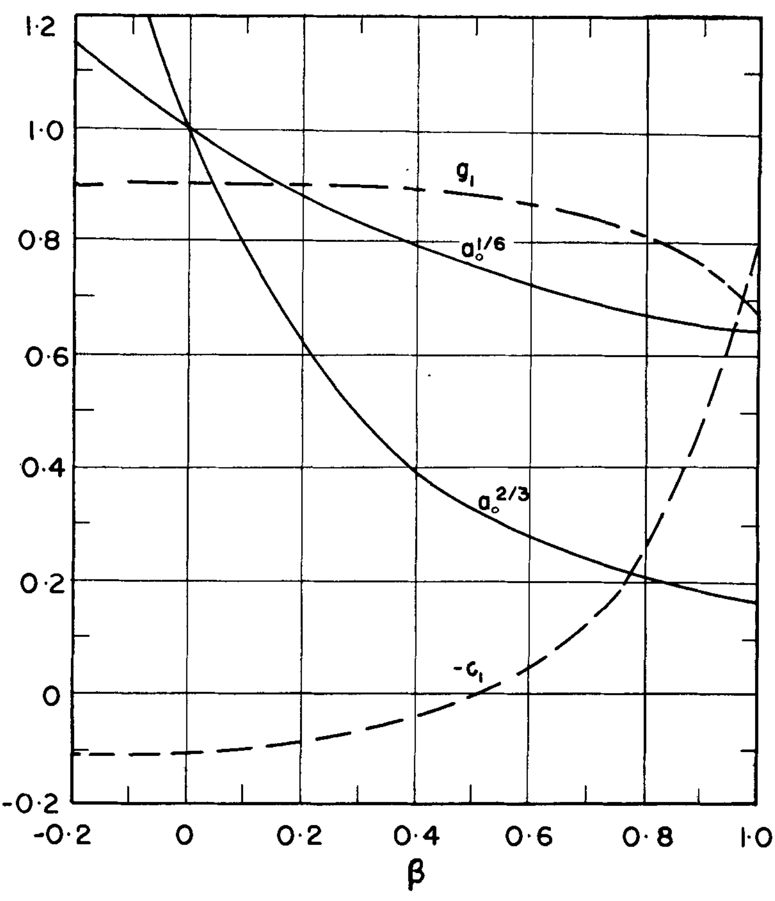

Figure 4. A plot of various quantities appearing in the numerical evaluation of equation (14).

it is necessary first of all to ascertain the value of κ^{-1} which satisfies the boundary conditions, $R_0 = R_1$, where these quantities are given in equations (16) and (12), respectively. In these expressions u_0 means the value of u at $y = 0$, i.e. at $\epsilon = \kappa^{-1}-1$, and u_1 is the value of u at $y = L$

or $\eta = \kappa^{-1} - 1 - M_1$. Having obtained κ^{-1} by a method of successive approximation, one then has β_1 and hence the various coefficients in the above power expansions, as functions of y, and, in terms of these, the function F is determined.

Figure 5 is a plot of the turbulent boundary layer profile $M/M_1 = (y/L)^{1/7}$ as a function of y/L. In figure 6 the sound pressure variation across such

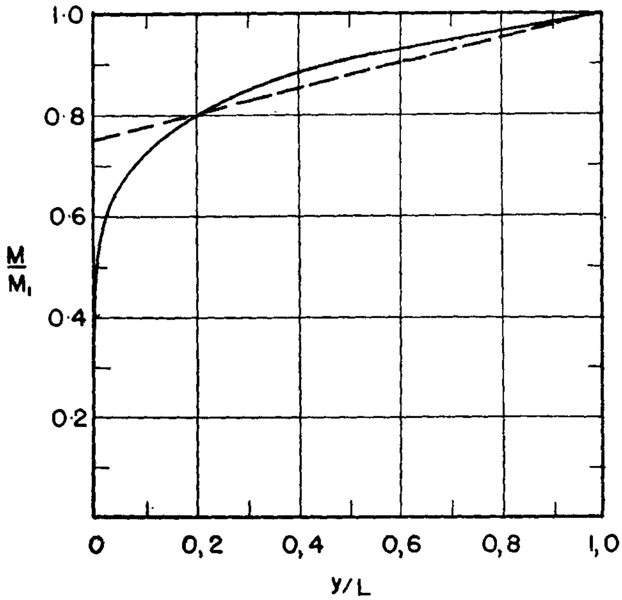

Figure 5. Turbulent boundary layer profile $M/M_1 = (y/L)^{1/7}$ as a function of y/L.

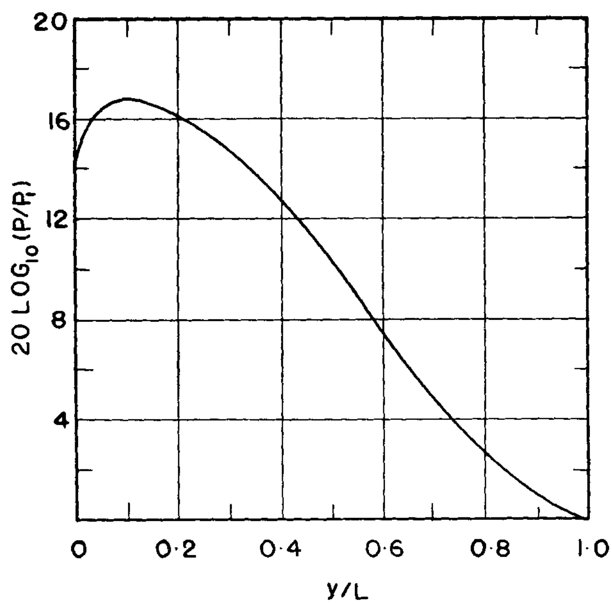

Figure 6. Sound pressure variation across the flow profile of figure 5 for $M_1 = 0{\cdot}2$ and $kL = 20$.

a profile is plotted for the case $M_1 = 0{\cdot}2$, $kL = 20$. The dip in the curve near $y = 0$ does not have a physical basis but is merely a manifestation of the fact that the Langer solution breaks down at this point. The curve is qualitatively similar to that for $M_1 = 0{\cdot}05$ in figure 2, which was drawn for the case of a constant flow gradient. This is physically not surprising if we consider the shape of the turbulent boundary-layer profile. Referring to figure 5 we see that the one-seventh-power curve can be replaced with a fair degree of approximation over most of its range by a straight line of slope $M_1/4L$ starting from $M = \frac{3}{4}M_1$ at $y = 0$, as is indicated by the dotted line. The area under the two curves is the same, although the dotted line involves a change in M of only $\frac{1}{4}M_1$. Thus, we see that a reasonable estimate of the sound pressure distribution across a turbulent shear layer is obtained by considering the distribution across a suitable constant flow gradient, for which the calculation is far less laborious.

5. Effect of flow on the attenuation of sound in a duct

We now suppose the side walls to be absorbing, so that the sound is attenuated as it propagates, and we ask what effect the presence of an air flow will have on this attenuation. We assume that the absorption of acoustic energy by the walls is sufficiently small so that the sound pressure profile across the duct is not sensibly different from that which has already been calculated on the assumption that the walls were non-absorbing. Then the power transmitted down a unit width of the duct in one of the boundary layers is

$$W = \int_0^L I(y)\,dy,$$

where $I(y)$ is the intensity of the sound field in the absence of absorption. On the other hand, the power absorbed in an interval dx of the side wall will be approximately

$$dW = -\left|\frac{p_0}{Z}\right|^2 \mathscr{R}(Z)\,dx$$
$$= -\frac{p_0^2}{\rho_0 c}\,\alpha\,dx.$$

Here p_0 is the value of p at the wall $y = 0$; $\mathscr{R}(Z)$ denotes the real part of the wall impedance Z, and $\alpha/\rho_0 c = \mathscr{R}(Z)/|Z|^2$ is the real part of the wall admittance. Thus the power in the duct diminishes according to the relation

$$W = W_0 \exp\left\{-\frac{p_0^2\,\alpha x}{\rho_0 c \int I(y)\,dy}\right\}. \tag{17}$$

If there were no flow and the acoustic pressure were uniform across the face of the duct, then we should have $I = p_0^2/(\rho_0 c)$ and

$$W = W_0\,e^{-\alpha x/L}.$$

This suggests writing (17) in the form

$$W = W_0\,e^{-n\alpha x/L},$$

where

$$n = \frac{p_0^2 L}{\rho_0 c \int_0^L I(y)\,dy}. \tag{18}$$

The quantity $n = n(kL, M)$, which is a function of both the frequency and the flow velocity, is then a measure of the increase in attenuation due to the presence of the flow. In other words, if the acoustic intensity in a certain frequency component is attenuated at the rate of, say, 6 decibels in a unit distance without flow, then in the presence of the flow it should be attenuated by $6n$ decibels in the same distance.

In order to evaluate n we must first obtain an expression for the intensity $I(y)$. It is more difficult to determine the acoustic intensity in the shear flow than it is to obtain the pressure profile across it owing to the fact that, unlike the pressure, the intensity is a second-order quantity (involving the square of the acoustic pressure). Hence, in deriving the intensity it would clearly be unjustified to neglect second-order terms in the basic equations as was done in linearizing equations (1) to (3).

In this paper we shall confine ourselves to deriving an expression for the acoustic intensity, correct to second order, for the one-dimensional case of a plane sound wave propagating in a medium which is moving with the (constant) velocity $V = cM$ in the direction of propagation. This is done in the Appendix, and the result turns out to be

$$I = \frac{p^2}{\rho_0 c}(1+M). \tag{19}$$

We shall then make the assumption that this expression represents a reasonable approximation to the actual intensity $I(y)$ in the shear flow if p and M are replaced by $p(y)$ and $M(y)$, i.e. by the known values of these quantities in the shear flow. This approximation is presumably better the higher the frequency.

On introducing $I(y)$ from (19) into (18), we obtain

$$\begin{aligned} n^{-1} &= \int_0^L \left(\frac{p}{p_0}\right)^2 (1+M)\frac{dy}{L} \\ &= \int_0^L \left(\frac{F}{F_0}\right)^2 (1+M)\frac{dy}{L}. \end{aligned} \tag{20}$$

In accordance with the discussion of the one-seventh-power profile of figure 5 given at the end of the last section, we suppose the Mach number of the flow to start from a value $M_0 = \frac{3}{4}M_1$ at $y = 0$ and to increase uniformly to M_1 at $y = L$. Now the pressure variation across the shear layer, given by F, depends on the velocity gradient only and is not changed by the superposition of the constant flow M_0. Thus the function F is still given by equation (9). On the other hand, the factor $(1+M)$ in equation (20) clearly involves the total Mach number. Thus, to the first order in $\epsilon = \eta - 1$ we have

$$F = 3^{-1/6}(1+0{\cdot}90\epsilon)f(u),$$

where $$f(u) = Bi(-u) - R\,Ai(-u),$$
and $$u = 2^{1/3}(\Lambda\kappa)^{2/3}\epsilon(1+0{\cdot}1\epsilon).$$
Also $$\frac{dy}{L} = -\frac{1}{k\kappa L}\left(\frac{\Lambda u}{2}\right)^{1/3}(1-\epsilon/5)\,du,$$
$$M = M_0 + \kappa^{-1} - \eta.$$

Substituting these expressions into equation (20) gives

$$n^{-1} = 3(\kappa^{-1}-1)\frac{1+M_0}{M_0}\frac{1}{f_0^2 u_0}\int_{u_1}^{u_0}(1+\Gamma u)\,du \tag{21}$$

where $$\Gamma = \left(\frac{8}{5} - \frac{1}{1+M_0}\right)2^{-1/3}(\Lambda\kappa)^{-2/3}.$$

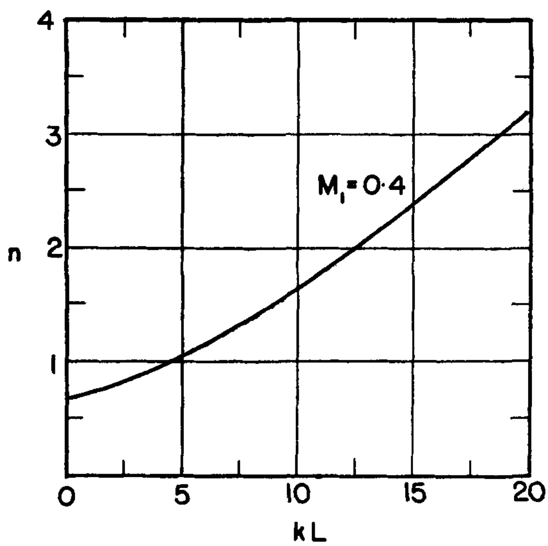

Figure 7. A plot of the attenuation parameter n as a function of kL for $M_1 = 0{\cdot}4$. The flow velocity increases linearly from a Mach number 0·3 at the wall to 0·4 at a distance L from the wall.

All quantities appearing in equation (21) are obtained from the results of the first section applied to a Mach number $\frac{1}{4}M_1$. The integral in (21) can be evaluated with the help of the following relations

$$\int f^2(u)\,du = uf^2 + f'^2,$$

$$\int uf^2(u)\,du = \tfrac{1}{3}(uf'^2 - ff' + u^2f^2),$$

which can be established directly from the equation (9a) satisfied by f.

In figure 7 we have plotted n *vs* kL for a Mach number of 0·4. It is clear from the figure that the effect of the flow is to increase the attenuation of the higher frequencies ($kL > 5$) but to diminish that of the lower frequencies ($kL < 5$). In figure 8 we have plotted n *vs* the Mach number for $kL = 2\pi$ and $kL = 20$. Here we see that for the higher frequency n increases with Mach number up to a certain point but then begins to diminish again when the Mach number becomes too large. On the other hand, for the longer wavelength ($kL = 2\pi$) n remains always near to unity, indicating that for this case the flow practically annuls the attenuation. As pointed out in the Introduction, these results are due to the opposing effects of refraction by the flow gradient and increased intensity due to the mean flow. The presence of these two factors is clear from the expression (20) for n where the term $(F/F_0)^2$ represents the contribution from refraction and the term $1+M$ that from the mean flow.

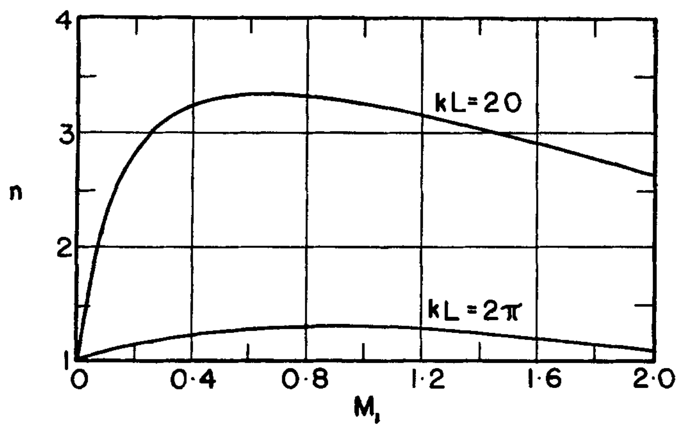

Figure 8. A plot of n *vs* M_1 for $kL = 2\pi$ and $kL = 20$. The flow increases linearly from $\frac{3}{4}M_1$ at the wall to M_1 at a distance L from the wall.

It should again be pointed out that the analysis in §3 has been applied only to subsonic Mach numbers. Thus, for example, the point $M_1 = 1$ on the abscissa in figure 8 is to be interpreted as meaning a constant flow with $M_0 = 0{\cdot}75$ superposed on a shear flow which increases linearly from $M = 0$ to $M = 0{\cdot}25$; §3 has been required only in dealing with the superposed shear flow.

The author is indebted to Professor Lighthill for several helpful discussions of this problem.

Appendix

Acoustic intensity in a moving medium

The energy flow in a moving medium is given by (cf. Blokhintsev 1956, p. 4)

$$\mathbf{N} = (\tfrac{1}{2}\rho v^2 + \rho E + p)\mathbf{v}, \tag{1 A}$$

where ρ is the density, E the internal energy per unit mass, p the pressure, and $\mathbf{v}$ the total velocity. For an ideal gas $\rho E = p/(\gamma - 1)$ where γ is the ratio of the specific heats. We now write

$$p = p_0 + p_1 + p_2, \qquad \rho = \rho_0 + \rho_1 + \rho_2, \qquad \mathbf{v} = \mathbf{V} + \mathbf{v}_1 + \mathbf{v}_2,$$

where p_0 is the undisturbed pressure, p_1 the first-order acoustic pressure, and p_2 the second-order acoustic pressure (of the order of p_1^2) and so on.

If we define the acoustic intensity I as the difference between the flow vectors $\mathbf{N}$ as calculated from (1 A) with and without the sound field present, we obtain with an accuracy up to terms of second order

$$I = \overline{N-N_0} = \tfrac{3}{2}\rho_0 v_1^2 V + \tfrac{3}{2}\rho_0 V^2 v_2 + \tfrac{3}{2}\rho_1 v_1 V^2 + \tfrac{1}{2}\rho_2 V^3 + \\ + \{\gamma/(\gamma-1)\}(p_2 V + p_1 v_1 + p_0 v_2), \quad (2\,\mathrm{A})$$

where the bar over $N-N_0$ indicates a time average, but bars are omitted from terms on the right for convenience. In deriving the above expression it has been assumed that the velocities $\mathbf{V}$, $\mathbf{v}_1$ and $\mathbf{v}_2$ all lie in the same direction and that the time average of all first-order quantities is zero.

We next express all second-order quantities in terms of first-order quantities as follows. The equation of conservation of mass in one dimension

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0$$

becomes on time-averaging

$$\rho_0 v_2 + \rho_1 v_1 + \rho_2 V = 0. \quad (3\,\mathrm{A})$$

Similarly the equation of conservation of momentum

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial}{\partial x}(\rho v^2) = -\frac{\partial p}{\partial x}$$

gives

$$\rho_0 v_1^2 + 2\rho_0 V v_2 + 2\rho_1 v_1 V + \rho_2 V^2 + p_2 = 0. \quad (4\,\mathrm{A})$$

Finally, the equation of state, $p = p_0(\rho/\rho_0)^\gamma$, gives

$$p_1 = c^2\rho_1, \quad (5\,\mathrm{A})$$

and

$$p_2 = c^2\rho_2 + \frac{\gamma-1}{2\rho_0}c^2\rho_1^2, \quad (6\,\mathrm{A})$$

where $c^2 = \gamma p_0/\rho_0$.

The three equations (3 A), (4 A) and (6 A) determine p_2, ρ_2 and v_2 in terms of zero- and first-order quantities;

$$p_2 = -\rho_0 v_1^2 \frac{2+M^2(\gamma-1)}{2(1-M^2)},$$

$$\rho_2 = -\rho_0 \frac{v_1^2}{c^2}\frac{\gamma+1}{2(1-M^2)},$$

$$v_2 = -\frac{\rho_1 v_1}{\rho_0}\left[1 - \frac{\gamma+1}{2}\frac{M}{1-M^2}\right],$$

where $M = V/c$.

If we now substitute these expressions into (2 A) and make use of (5 A), together with the plane wave relation $p_1 = \rho_0 c v_1$, we obtain, after reduction,

$$I = \frac{p_1^2}{\rho_0 c}(1+M). \quad (7\,\mathrm{A})$$

References

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